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Schrödinger–Dirac Spaces of Entire Functions

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A study is made of some Hilbert spaces of entire functions which appear in the quantum mechanical theory of the hydrogen atom.

The spaces are examples in the theory [1] of Hilbert spaces whose elements are entire functions and which have these properties:

(H1) Whenever $F(z)$ is in the space and has a nonreal zero w, the function $F(z)(z - \bar{w})/(z - w)$ belongs to the space and has the same norm as $F(z)$.

(H2) For each nonreal number w, the linear functional defined on the space by taking $F(z)$ into $F(w)$ is continuous.

(H3) The function $F^*(z) = \bar{F}(\bar{z})$ belongs to the space whenever $F(z)$ belongs to the space and it always has the same norm as $F(z)$.

The theory of these spaces is related to the theory of entire functions $E(z)$ which satisfy the inequality $|E(x - iy)| < |E(x + iy)|$ for $y > 0$. If $E(z)$ is any such function, write $E(z) = A(z) - iB(z)$, where $A(z)$ and $B(z)$ are entire functions which are real for real z and

$$K(w, z) = [B(z)\,\bar{A}(w) - A(z)\,\bar{B}(w)]/[\pi(z - \bar{w})].$$

Then the set $\mathscr{H}(E)$ of entire functions $F(z)$ such that

$$\|F\|^2 = \int_{-\infty}^{+\infty} |F(t)/E(t)|^2\, dt < \infty$$

and such that

$$|F(z)|^2 \leqslant \|F\|^2 K(z, z)$$

for all complex z is a Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3). For each complex number w, $K(w, z)$ belongs to the space as a function of z and the identity

$$F(w) = \langle F(t), K(w, t)\rangle$$

holds for every element $F(z)$ of the space. A Hilbert space whose elements are

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entire functions, which satisfies the axioms (H1), (H2), and (H3), and which contains a nonzero element, is isometrically equal to a space $\mathscr{H}(E)$.

Such spaces appear in the eigenfunction expansions associated with first- and second-order differential operators. The spaces now studied appear in the expansion of the operator

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{d}{dr} + \begin{pmatrix} a & b \\ b & -a \end{pmatrix} + \begin{pmatrix} s+p & q \\ q & s-p \end{pmatrix} k \cot(kr)$$

in the space of square integrable functions of r, $0 < r < \pi/k$. The parameters a, b, p, q, and s are real numbers such that

$$p^2 + q^2 \leqslant s^2 \qquad \text{and} \qquad k^2(p^2 + q^2) \leqslant a^2 + b^2 + k^2 s^2,$$

and k is positive or zero. In the classical theories of the hydrogen atom, k is zero. The case k positive corresponds to a generalization of the theories in a space of constant positive curvature.

A useful notation is suggested by the quantum mechanical application. Define mass M as the nonnegative solution of the equation

$$M^2 = a^2 + b^2 + k^2 s^2 - k^2 p^2 - k^2 q^2$$

and charge Q by

$$Q = ap + bq.$$

Angular momentum n is the nonnegative solution of the equation

$$n^2 = p^2 + q^2 - s^2.$$

Only the case of integral n is now considered.

The classical treatment of the hydrogen atom consists of two separate theories, the initial one due to Schrödinger and its variant due to Dirac. The parameter s vanishes in the Schrödinger theory and has the value $-Q/M$ in the Dirac theory. Since the parameter is now unrestricted, the expansion obtained interpolates between the classical expansions. The number s, which is a basic invariant like mass and charge, will be called the Dirac number.

Two Schrödinger–Dirac operators

$$H_\pm = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{d}{dr} + \begin{pmatrix} a_\pm & b_\pm \\ c_\pm & d_\pm \end{pmatrix} + \begin{pmatrix} s_\pm + p_\pm & q_\pm \\ q_\pm & s_\pm - p_\pm \end{pmatrix} k \cot(kr$$

are taken to be equivalent if the equation

$$H_+ \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} H_-$$

is soluble for a real number θ. The eigenfunction expansions of equivalent Schrödinger–Dirac operators are obtained from each other by a change of variable. Equivalent Schrödinger–Dirac operators have equal mass, charge, Dirac number, and angular momentum.

A Schrödinger–Dirac operator is said to be in normal form if $p = s$. Every Schrödinger–Dirac operator is equivalent to one which is in normal form. The normal form is unique in the case of zero angular momentum. Two Schrödinger–Dirac operators of zero angular momentum are equivalent if they have equal mass, charge, and Dirac number. When the angular momentum is positive, there are two normal forms, a positive normal form in which $q = n$ and a negative normal form in which $q = -n$. The coefficients $-a_n$ and $-a_{-n}$ in these normal forms are the two solutions of the quadratic equation

$$(n^2 + s^2)\lambda^2 + 2Qs\lambda + Q^2 = M^2n^2 + k^2n^4$$

in λ. These numbers are eigenvalues of the operator, called its central eigenvalues. Two Schrödinger–Dirac operators of equal mass, charge, Dirac number, and angular momentum are equivalent if the central eigenvalues in their negative normal forms are equal.

Relations exist between Schrödinger–Dirac operators which have equal mass, charge, and Dirac number but whose angular momenta differ by 1.

THEOREM 1. *If*

$$H_{-n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{d}{dr} + \begin{pmatrix} a_{-n} & b_{-n} \\ b_{-n} & -a_{-n} \end{pmatrix} + \begin{pmatrix} s + p_{-n} & q_{-n} \\ q_{-n} & s - p_{-n} \end{pmatrix} k \cot(kr)$$

is a Schrödinger–Dirac operator of mass M, charge Q, Dirac number s, and angular momentum n in negative normal form and if

$$H_{n+1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{d}{dr} + \begin{pmatrix} a_{n+1} & b_{n+1} \\ b_{n+1} & -a_{n+1} \end{pmatrix} + \begin{pmatrix} s + p_{n+1} & q_{n+1} \\ q_{n+1} & s - p_{n+1} \end{pmatrix} k \cot(kr)$$

is a Schrödinger–Dirac operator of mass M, charge Q, Dirac number s, and angular momentum $n + 1$ in positive normal form, then the identities

$$H_{-n}W_{-n} = W_{-n}H_{n+1} \qquad \textit{and} \qquad W_nH_{-n} = H_{n+1}W_n$$

hold with

$$2W_{-n} = H_{-n} + H_{n+1} + \begin{pmatrix} a_{-n} & -b_{-n} \\ b_{-n} & a_{-n} \end{pmatrix} + \begin{pmatrix} a_{n+1} & b_{n+1} \\ -b_{n+1} & a_{n+1} \end{pmatrix} + (2n+1) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} k \cot(kr)$$

and

$$2W_n = H_{-n} + H_{n+1}$$
$$+ \begin{pmatrix} a_{-n} & b_{-n} \\ -b_{-n} & a_{-n} \end{pmatrix} + \begin{pmatrix} a_{n+1} & -b_{n+1} \\ b_{n+1} & a_{n+1} \end{pmatrix} + (2n+1) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} k \cot(kr).$$

The identities

$$W_{-n} W_n = (H_{-n} - \lambda_{-n})(H_{-n} - \lambda_{n+1}),$$

and

$$W_n W_{-n} = (H_{n+1} - \lambda_{-n})(H_{n+1} - \lambda_{n+1})$$

hold with λ_{-n} and λ_{n+1} equal to the central eigenvalues of H_{-n} and H_{n+1}.

These relations are iterated in obtaining properties of the expansion. For any given mass M, charge Q, and Dirac number s, the integers n for which a Schrödinger–Dirac operator of angular momentum n exists with these invariants are restricted by the inequality

$$Q^2 \leqslant (M^2 + k^2 n^2)(s^2 + n^2).$$

The inequality is satisfied for every positive integer n in the Schrödinger theory. In the Dirac theory it is also satisfied when it is zero.

If n is a positive integer and if H_n and H_{-n} are equivalent Schrödinger–Dirac operators of angular momentum n, H_n in positive normal form and H_{-n} in negative normal form, then

$$H_{-n} \begin{pmatrix} s & n \\ -n & s \end{pmatrix} = \begin{pmatrix} s & n \\ -n & s \end{pmatrix} H_n .$$

A normalization of eigenfunctions is needed in eigenfunction expansions. If H_n is a Schrödinger–Dirac operator of positive angular momentum n in positive normal form, let $A_n(r, w)$ and $B_n(r, w)$ be the unique differentiable functions of r, $0 < r < \pi/k$, for each complex number w such that

$$H_n \begin{pmatrix} A_n(r, w) \\ B_n(r, w) \end{pmatrix} = w \begin{pmatrix} A_n(r, w) \\ B_n(r, w) \end{pmatrix},$$

such that

$$1 \times 3 \times \cdots \times (2n-1) \lim_{r \searrow 0} A_n(r, w)/r^n = 1,$$

and such that

$$1 \times 3 \times \cdots \times (2n-1) \lim_{r \searrow 0} B_n(r, w)/r^n = -s/n.$$

If H_{-n} is a Schrödinger–Dirac operator of angular momentum n in negative normal form and if λ_{-n} is its central eigenvalue, let $A_{-n}(r, w)$ and $B_{-n}(r, w)$ be the unique differentiable functions of r, $0 < r < \pi/k$, for each complex number w such that

$$H_{-n}\begin{pmatrix} A_{-n}(r, w) \\ B_{-n}(r, w) \end{pmatrix} = w \begin{pmatrix} A_{-n}(r, w) \\ B_{-n}(r, w) \end{pmatrix},$$

such that

$$1 \times 3 \times \cdots \times (2n+1) \lim_{r \searrow 0} A_{-n}(r, w)/r^{n+1} = w - \lambda_{-n},$$

and such that

$$1 \times 3 \times \cdots \times (2n+1) \lim_{r \searrow 0} B_{-n}(r, w)/r^{n} = -(2n+1).$$

When n is positive, these normalizations imply that the relations

$$A_n(r, w) = -B_{-n}(r, w) - s/n\, A_{-n}(r, w)$$

and

$$B_n(r, w) = A_{-n}(r, w) - s/n\, B_{-n}(r, w)$$

hold when H_n in positive normal form and H_{-n} in negative normal form are equivalent.

The identities

$$A_{-n}(r, w) = (w - \lambda_{-n})\, A_{n+1}(r, w)$$

and

$$\begin{aligned} B_{-n}(r, w) &= (w - \lambda_{n+1})\, B_{n+1}(r, w) \\ &\qquad - [b_{n+1} - b_{-n} + (2n+1)\, k \cot(kr)]\, A_{n+1}(r, w) \end{aligned}$$

hold for every nonnegative integer n when H_{-n} and H_{n+1} have equal mass, charge, and Dirac number, H_{-n} has angular momentum n and is in negative normal form, H_{n+1} has angular momentum $n+1$ and is in positive normal form, and λ_{-n} and λ_{n+1} are the central eigenvalues of H_{-n} and H_{n+1}.

Note that if H_{-n} is a Schrödinger–Dirac operator of angular momentum n in negative normal form, then

$$A_{-n}(r, \lambda_{-n}) = 0$$

and

$$1 \times 3 \times \cdots \times (2n+1)\, B_{-n}(r, \lambda_{-n}) = -(2n+1) \exp(-b_{-n} r) \sin^n(kr)/k^n,$$

where λ_{-n} is the central eigenvalue of H_{-n}.

An eigenfunction expansion follows.

THEOREM 2. *If H is a Schrödinger–Dirac operator in normal form, then a space $\mathcal{H}(E(a))$ exists, $E(a, z) = A(a, z) - iB(a, z)$, for each number a, $0 < a < \pi/k$. A necessary and sufficient condition that an entire function $F(z)$ belong to the space is that*

$$\pi F(z) = \int_0^a f_+(r)\, A(r, z)\, dr + \int_0^a f_-(r)\, B(r, z)\, dr$$

for square integrable functions $f_+(r)$ and $f_-(r)$ which vanish outside of the interval $(0, a)$. The identity

$$\pi \,\|F(t)\|^2 = \int_0^a |f_+(r)|^2\, dr + \int_0^a |f_-(r)|^2\, dr$$

is then satisfied. If $F(z)$ and $G(z)$ are elements of $\mathcal{H}(E(a))$ and if $f_+(r)$, $f_-(r)$, $g_+(r)$, and $g_-(r)$ are the corresponding square integrable functions of r, then the condition $G(z) = zF(z)$ is equivalent to the equations

$$\begin{aligned} g_+(r) = f_-'(r) &+ [a + 2sk \cot(kr)]\, f_+(r) \\ &+ [b + qk \cot(kr)]\, f_-(r) \end{aligned}$$

and

$$g_-(r) = -f_+'(r) + [b + qk \cot(kr)]\, f_+(r) - af_-(r)$$

with the boundary condition $f_+(0) = 0$ if $n = 0$.

When the angular momentum is positive, the eigenfunction expansions corresponding to the normal forms are obtained from each other by a linear change of variable.

THEOREM 3. *If H_{-n} and H_n are equivalent Schrödinger–Dirac operators of positive angular momentum n, H_n in positive normal form and H_{-n} in negative normal form, then $\mathcal{H}(E_{-n}(a))$ and $\mathcal{H}(E_n(a))$ coincide as sets and the identity*

$$\|F(t)\|^2_{E_{-n}(a)} = (1 + s^2/n^2)\, \|F(t)\|^2_{E_n(a)}$$

holds for every element $F(z)$ of these spaces. If $F(z)$ is an element of $\mathcal{H}(E_{-n}(a))$, if $G(z)$ is an element of $\mathcal{H}(E_n(a))$, and if $f_+(r)$, $f_-(r)$, $g_+(r)$, and $g_-(r)$ are the corresponding square integrable functions of r, then the condition $F(z) = G(z)$ is equivalent to the conditions

$$f_+(r) = g_-(r) - s/n\, g_+(r)$$

and

$$f_-(r) = -g_+(r) - s/n\, g_-(r).$$

Relations exist between spaces appearing in eigenfunction expansions associated with Schrödinger–Dirac operators which have equal mass, charge, and Dirac number but whose angular momenta differ by 1.

THEOREM 4. *Let H_{-n} and H_{n+1} be Schrödinger–Dirac operators of equal mass, charge, and Dirac number, H_{-n} of angular momentum n in negative normal form and H_{n+1} of angular momentum $n+1$ in positive normal form, and let λ_{-n} and λ_{n+1} be their central eigenvalues. Then for each parameter a, $0 < a < \pi/k$, $\mathscr{H}(E_{n+1}(a))$ coincides with the domain of multiplication by z in $\mathscr{H}(E_{-n}(a))$ and the identity*

$$\langle F(t), G(t)\rangle_{E_{n+1}(a)} = \langle (t-\lambda_{-n}) F(t), (t-\lambda_{n+1})\, G(t)\rangle_{E_{-n}(a)}$$

holds for all elements $F(z)$ and $G(z)$ of $\mathscr{H}(E_{n+1}(a))$. If $F(z)$ is an element of $\mathscr{H}(E_{-n}(a))$, if $G(z)$ is an element of $\mathscr{H}(E_{n+1}(a))$, and if $f_+(r), f_-(r), g_+(r)$, and $g_-(r)$ are the corresponding square integrable functions, the condition $F(z) = G(z)$ is equivalent to the condition that W_n take $\binom{f_+}{f_-}$ into $\binom{g_+}{g_-}$.

If H_{-n} is a Schrödinger–Dirac operator of mass M, charge Q, Dirac number s, and angular momentum n in negative normal form, and if λ_{-n} is its central eigenvalue, define the entire function $S_{-n}(z)$ by

$$\begin{aligned}(-1)^n\, n!n!k^{2n+1}S_{-n}(z)\\ = \pi(z-\lambda_{-n}) \prod_{m>n} [1-(z^2-M^2)/(k^2m^2)-(sz+Q)^2/(k^2m^4)]\end{aligned}$$

when k is positive. Note that $S_0'(\lambda_0) = b_0^{-1}\sinh(\pi b_0/k)$ and that

$$\begin{aligned}(-1)^n \frac{1\times 3\times\cdots\times(2n-1)}{2\times 4\times\cdots\times(2n)}\, S'_{-n}(\lambda_{-n})\\ = b_{-n}^{-1}\sinh(\pi b_{-n}/k)\Big/\prod_{m=1}^{n}(4b_{-n}^2+4k^2m^2)\end{aligned}$$

when $n > 0$. If the angular momentum n is positive and if H_n is the equivalent Schrödinger–Dirac operator in positive normal form, define

$$S_n(z) = (1+s^2/n^2)\, S_{-n}(z).$$

If H_{n+1} is a Schrödinger–Dirac operator of mass M, charge Q, Dirac number s, and angular momentum $n+1$ in positive normal form, and if λ_{n+1} is its central eigenvalue, then

$$S_{-n}(z) = (z-\lambda_{-n})(z-\lambda_{n+1})\, S_{n+1}(z).$$

Every zero of $S_{-n}(z)$ other than λ_{-n} is the central eigenvalue λ_m of a Schrödinger–Dirac operator H_m of mass M, charge Q, Dirac number s, and angular momentum m in positive normal form, or the central eigenvalue λ_{-m} of an equivalent Schrödinger–Dirac operator H_{-m} in negative normal form, for some integer m

greater than n. The coefficients b_m and b_{-m} are defined accordingly so that $m b_m = s\lambda_m + Q$ and $-mb_{-m} = s\lambda_{-m} + Q$.

An explicit description of the spaces follows in the case of positive curvature.

THEOREM 5. *If H_{-n} is a Schrödinger–Dirac operator in negative normal form and if k is positive, a necessary and sufficient condition that an entire function $F(z)$ belong to $\mathcal{H}(E_{-n}(a))$ is that it be of bounded type and of mean type at most a in the upper and lower half-planes and that*

$$\pi \sum_{m=n}^{\infty} (-1)^m \exp(\pi b_{-m}/k) \mid F(\lambda_{-m})|^2/S'_{-n}(\lambda_{-m})$$

$$+ \pi \sum_{m=n+1}^{\infty} (-1)^m \exp(-b_m/k) \mid F(\lambda_m)|^2/S_n'(\lambda_m) < \infty.$$

The sum is then equal to $\|F(t)\|^2_{E_{-n}(a)}$.

A similar result holds in the case of zero curvature despite the degeneracy of the product defining $S(z)$. For a Schrödinger–Dirac operator of angular momentum n in normal form, define

$$W(0) = \pi b_0 \exp(\pi b_0/k)/\sinh(\pi b_0/k)$$

if n is zero,

$$W(-n) = \frac{1 \times 3 \times \cdots \times (2n-1)}{2 \times 4 \times \cdots \times (2n)} \pi b_{-n} \exp(\pi b_{-n}/k)/\sinh(\pi b_{-n}/k)$$
$$\times \prod_{t=1}^{n} (4b_n{}^2 + 4k^2t^2)$$

if n is positive and the operator is in negative normal form, and

$$W(n) = \frac{1 \times 3 \times \cdots \times (2n-1)}{2 \times 4 \times \cdots \times (2n)} \pi b_n \exp(-\pi b_n/k)/\sinh(\pi b_n/k)$$
$$\times \prod_{t=1}^{n} (4b_n{}^2 + 4k^2t^2)$$

if n is positive and the operator is in positive normal form. For every integer m greater than n, define $W(-m, n)$ and $W(m, n)$ by

$$W(-m) = W(-m, n) \prod_{t=n+1}^{m} (1 + s^2/t^2)$$
$$\times (\lambda_{-m} - \lambda_{-n})(\lambda_{-m} - \lambda_{-n-1}) \cdots (\lambda_{-m} - \lambda_{-m+1})$$
$$\times (\lambda_{-m} - \lambda_{n+1})(\lambda_{-m} - \lambda_{n+2}) \cdots (\lambda_{-m} - \lambda_m)$$

and

$$W(m) = W(m, n) \prod_{t=n+1}^{m} (1 + s^2/t^2)$$
$$\times (\lambda_m - \lambda_{-n})(\lambda_m - \lambda_{-n-1}) \cdots (\lambda_m - \lambda_{-m})$$
$$\times (\lambda_m - \lambda_{n+1})(\lambda_m - \lambda_{n+2}) \cdots (\lambda_m - \lambda_{-m+1}).$$

Then

$$W(-m, n) = \pi(-1)^m \exp(\pi b_{-m}/k)/S'_{-n}(\lambda_{-m})$$

and

$$W(m, n) = \pi(-1)^m \exp(-\pi b_m/k)/S'_{-n}(\lambda_m)$$

when k is positive. The computation of norms in Theorem 5 then reads

$$\| F(t)\|^2_{E_{-n}(a)} = \sum_{m=n}^{\infty} W(-m, n) \mid F(\lambda_{-m})\mid^2 + \sum_{m=n+1}^{\infty} W(m, n) \mid F(\lambda_m)\mid^2.$$

An analogous result holds in the limit of small k.

THEOREM 6. *When k is zero in Theorem 5, a necessary and sufficient condition that an entire function $F(z)$ belongs to $\mathscr{H}(E_{-n}(a))$ is that it be of bounded type and of mean type at most a in the upper and lower half-planes and that*

$$\int_{-\infty}^{-M} \left| F(\lambda)\, \Gamma\left(1 + n + i \frac{s\lambda + Q}{(\lambda^2 - M^2)^{1/2}}\right) \Big/ \Gamma(1 + n) \right|^2$$
$$\times \exp\left(-\pi \frac{s\lambda + Q}{(\lambda^2 - M^2)^{1/2}}\right) (\lambda^2 - M^2)^{n+1} \frac{d\lambda}{\lambda(\lambda - \lambda_{-n})}$$
$$+ \int_{M}^{\infty} \left| F(\lambda)\, \Gamma\left(1 + n + i \frac{s\lambda + Q}{(\lambda^2 - M^2)^{1/2}}\right) \Big/ \Gamma(1 + n) \right|^2$$
$$\times \exp\left(-\pi \frac{s\lambda + Q}{(\lambda^2 - M^2)^{1/2}}\right) (\lambda^2 - M^2)^{n+1} \frac{d\lambda}{\lambda(\lambda - \lambda_{-n})}$$
$$+ \sum_{m=n}^{\infty} W(-m, n) \mid F(\lambda_{-m})\mid^2 + \sum_{m=n+1}^{\infty} W(m, n) \mid F(\lambda_m)\mid^2 < \infty.$$

In this case the sum is equal to $\| F(t)\|^2_{E_{-n}(a)}$.

These expansions appear in a generalization of the Schrödinger and Dirac theories of the hydrogen atom. If k is a nonnegative number, consider the three-dimensional space of constant curvature which has the metric element

$$dr^2 + \sin^2(kr)/k^2\, d\theta^2 + \sin^2(kr)/k^2 \sin^2(\theta)\, d\varphi^2,$$

where $0 < r < \pi/k$ and where spherical coordinates are defined by

$$x = r\sin(\theta)\cos(\varphi), \qquad y = r\sin(\theta)\sin(\varphi), \qquad z = r\cos(\theta).$$

All theories of the hydrogen atom are generalizations of the electromagnetic field theory [1]. An electromagnetic field is described by four quantities, Φ, E, H, Ψ, of which Φ and Ψ are measurable complex valued functions of position called the electric and magnetic potentials and

$$E = \mathbf{i}E_x + \mathbf{j}E_y + \mathbf{k}E_z \qquad \text{and} \qquad H = \mathbf{i}H_x + \mathbf{j}H_y + \mathbf{k}H_z$$

are measurable vector valued functions of position called the electric and magnetic vectors. The energy of the field, defined as one-half of the integral

$$\iiint (\Phi\bar{\Phi} + E\cdot\bar{E} + H\cdot\bar{H} + \Psi\bar{\Psi})\sin(kr)/(kr)\,dx\,dy\,dz$$

taken over the whole space, is assumed to be finite.

The spin operators σ_x, σ_y, σ_z, defined by

$$\begin{aligned}
\sigma_x(\Phi, E, H, \Psi) &= (-\mathbf{i}\cdot H, \mathbf{i}\times E - \mathbf{i}\Psi, \mathbf{i}\times H + \mathbf{i}\Phi, \mathbf{i}\cdot E),\\
\sigma_y(\Phi, E, H, \Psi) &= (-\mathbf{j}\cdot H, \mathbf{j}\times E - \mathbf{j}\Psi, \mathbf{j}\times H + \mathbf{j}\Phi, \mathbf{j}\cdot E),\\
\sigma_z(\Phi, E, H, \Psi) &= (-\mathbf{k}\cdot H, \mathbf{k}\times E - \mathbf{k}\Psi, \mathbf{k}\times H + \mathbf{k}\Phi, \mathbf{k}\cdot E)
\end{aligned}$$

satisfy the identities

$$\begin{aligned}
\sigma_x &= \sigma_y\sigma_z = -\sigma_z\sigma_y,\\
\sigma_y &= \sigma_z\sigma_x = -\sigma_x\sigma_z,\\
\sigma_z &= \sigma_x\sigma_y = -\sigma_y\sigma_x,\\
-1 &= \sigma_x^2 = \sigma_y^2 = \sigma_z^2.
\end{aligned}$$

The operators

$$\begin{aligned}
S_x &= x/r\,\partial/\partial r + k\cot(kr)(r\,\partial/\partial x - x\,\partial/\partial r) - \tfrac{1}{2}k\sigma_x,\\
S_y &= y/r\,\partial/\partial r + k\cot(kr)(r\,\partial/\partial y - y\,\partial/\partial r) - \tfrac{1}{2}k\sigma_y,\\
S_z &= z/r\,\partial/\partial r + k\cot(kr)(r\,\partial/\partial z - z\,\partial/\partial r) - \tfrac{1}{2}k\sigma_z
\end{aligned}$$

and

$$\begin{aligned}
L_x &= y\,\partial/\partial z - z\,\partial/\partial y + \tfrac{1}{2}\sigma_x,\\
L_y &= z\,\partial/\partial x - x\,\partial/\partial z + \tfrac{1}{2}\sigma_y,\\
L_z &= x\,\partial/\partial y - y\,\partial/\partial x + \tfrac{1}{2}\sigma_z
\end{aligned}$$

satisfy the identities

$$\begin{array}{lll} L_zL_y - L_yL_z = L_x\,, & L_xL_z - L_zL_x = L_y\,, & L_yL_x - L_xL_y = L_z\,, \\ S_zS_y - S_yS_z = k^2L_x\,, & S_xS_z - S_zS_x = k^2L_y\,, & S_yS_x - S_xS_y = k^2L_z\,, \\ L_zS_y - S_yL_z = S_x\,, & L_xS_z - S_zL_x = S_y\,, & L_yS_x - S_xL_y = S_z\,, \\ S_zL_y - L_yS_z = S_x\,, & S_xL_z - L_zS_x = S_y\,, & S_yL_x - L_xS_y = S_z\,, \\ L_xS_x - S_xL_x = 0, & L_yS_y - S_yL_y = 0, & L_zS_z - S_zL_z = 0. \end{array}$$

The operators

$$\sigma_r = (\sigma_x x + \sigma_y y + \sigma_z z)/r$$

and

$$D = 1 + \sigma_x(y\,\partial/\partial z - z\,\partial/\partial y) + \sigma_y(z\,\partial/\partial x - x\,\partial/\partial y) + \sigma_z(x\,\partial/\partial y - y\,\partial/\partial x)$$

anticommute. The operator

$$\begin{aligned} kD &+ \sigma_x[x/r\,\partial/\partial r + k\cot(kr)(r\,\partial/\partial x - x\,\partial/\partial r)] \\ &+ \sigma_y[y/r\,\partial/\partial r + k\cot(kr)(r\,\partial/\partial y - y\,\partial/\partial r)] \\ &+ \sigma_z[z/r\,\partial/\partial r + k\cot(kr)(r\,\partial/\partial z - z\,\partial/\partial r)] \\ &= \sigma_r[\partial/\partial r + k\cot(kr)] - k\cot(kr)\,\sigma_r D + kD \end{aligned}$$

commutes with S_x, S_y, S_z and L_x, L_y, L_z.

The operators I and J defined by

$$I(\Phi, E, H, \Psi) = (\Psi, H, -E, -\Phi)$$

and

$$J(\Phi, E, H, \Psi) = (\Phi, -E, H, -\Psi)$$

commute with the spin operators and anticommute with each other.

Maxwell's equations for the propagation of the electromagnetic field can be written

$$(1/c)(\partial/\partial t)(\Phi, E, H, \Psi) = K(\Phi, E, H, \Psi)$$

where c is the speed of light and K is the operator

$$K = I\sigma_r[\partial/\partial r + k\cot(kr)] - k\cot(kr)\,I\sigma_r D + kID.$$

Hydrogen theories are obtained when K is replaced by the operator

$$\begin{aligned} K = I\sigma_r[\partial/\partial r + k\cot(kr)] - k\cot(kr)\,I\sigma_r D + kID \\ + i\alpha J + i\beta IJ + isk\cot(kr) + isJDk\cot(kr) \end{aligned}$$

for real numbers α, β, and s. The mass M is $(\alpha^2 + \beta^2)^{1/2}$, the charge Q is $s\alpha$, and the Dirac number is s. Angular momentum enters on separation of variables in spherical coordinates, using the fact that D^2 commutes with K and that the eigenvalues of D are integers. See the related calculations for the Dirac theory [1].

These hydrogen theories are equivalent to the Dirac theory in the case that β and k vanish. The theories with β and k nonzero are suggested as being of equal physical interest.

Proof of Theorem 1. These identities are verified by straightforward calculations.

Proof of Theorem 2. The expansion is obtained by change of variable from the expansion theorem for Hilbert spaces of entire functions [1, Theorem 44], and the related [1, Theorem 45].

Proof of Theorem 3. The theorem follows from Theorem 2.

Proof of Theorem 4. By [1, Problem 289], the identities

$$A_{-n}(a, z) = (z - \lambda_{-n})\, A_{n+1}(a, z)$$

and

$$\begin{aligned} B_{-n}(a, z) = {} & (z - \lambda_{n+1})\, B_{n+1}(a, z) \\ & - [b_{n+1} - b_{-n} + (2n + 1)\, k \cot(kr)]\, A_{n+1}(a, z) \end{aligned}$$

imply that the space $\mathscr{H}(E_{n+1}(a))$ coincides with the domain of multiplication by z in $\mathscr{H}(E_{-n}(a))$ and that the stated relation holds between inner products.

The same identities imply that W_{-n} takes

$$\begin{pmatrix} A_{n+1}(r, w) \\ B_{n+1}(r, w) \end{pmatrix} \quad \text{into} \quad \begin{pmatrix} A_{-n}(r, w) \\ B_{-n}(r, w) \end{pmatrix}$$

for every complex number w. Assume that W_n takes $\binom{f_+}{f_-}$ into $\binom{g_+}{g_-}$ where $f_+(r)$ and $f_-(r)$ are the square integrable functions of r corresponding to an element $F(z)$ of $\mathscr{H}(E_{-n}(a))$ and $g_+(r)$ and $g_-(r)$ are the square integrable functions of r corresponding to an element $G(z)$ of $\mathscr{H}(E_{n+1}(a))$. A straightforward calculation will show that $F(w) = G(w)$. The theorem follows.

Proof of Theorem 5. Note that $(-1)^m S'_{-n}(\lambda_{-m})$ and $(-1)^m S'_{-n}(\lambda_m)$ are positive. By Theorem 4 and the theory of measures associated with Hilbert spaces of entire functions [1, Theorem 42], a nonnegative function $\rho(m)$ of integral m exists such that the identity

$$\begin{aligned} \|F(t)\|^2_{E_{-n}(a)} = {} & \pi \sum_{m=n}^{\infty} (-1)^m \exp(\pi b_{-m}/k)\, |F(\lambda_{-m})|^2\, \rho(-m)/S'_{-n}(\lambda_{-m}) \\ & + \pi \sum_{m=n+1}^{\infty} (-1)^m \exp(-\pi b_m/k)\, |F(\lambda_m)|^2\, \rho(m)/S'_{-n}(\lambda_m) \end{aligned}$$

holds for every element $F(z)$ of $\mathscr{H}(E_{-n}(a))$ independently of a. The function ρ remains unchanged when n is replaced by a larger positive integer. The theorem follows from [1, Theorem 39] once it is shown that $\rho(m)$ is identically one. By Theorems 2 and 4, it is sufficient to show that

$$\pi/W(-n) = \int_0^{\pi/k} |A_{-n}(r, \lambda_{-n})|^2 \, dr + \int_0^{\pi/k} |B_{-n}(r, \lambda_{-n})|^2 \, dr.$$

This is true because

$$\begin{aligned}&1^2 \times 3^2 \times \cdots \times (2n+1)^2 \, \pi/W(-n) \\ &\qquad = (2n+1)^2 \int_0^{\pi/k} \exp(-2b_{-n}r) \sin^{2n}(kr)/k^{2n} \, dr.\end{aligned}$$

Proof of Theorem 6. The theorem is obtained by passing to a limit in Theorem 5. The functions $E_{-n}(a, z)$, which depend implicitly on the parameter k, converge uniformly for z in any bounded set. By the theory of measures associated with Hilbert spaces of entire functions [1, Theorems 31, 32, 33], the measures associated with the limit spaces are obtained as limits of the measures associated with the approximating spaces. The discrete part of the spectrum of the limit measures, which lies in the interval $(-M, M)$, is easily computed because the mth eigenvalue λ_m depends continuously on k in the limit for each fixed m.

A more complicated calculation is needed to compute the continuous part of the spectrum, which lies in the half-lines $(-\infty, -M)$ and (M, ∞). The limiting behavior in this case is seen from the identity

$$\begin{aligned}&S'_{-n}(\lambda_m) \prod_{t=1}^{n} (\lambda_m{}^2 - M^2 - k^2t^2) \\ &\quad = (1 + s^2/m^2)(\lambda_m - \lambda_{-m})(\lambda_m - \lambda_{-n})/(\lambda_m{}^2 - M^2 - k^2m^2) \\ &\quad \times \sin(\pi/k(\lambda_m{}^2 - M^2)^{1/2})/(\lambda_m{}^2 - M^2)^{1/2} \\ &\quad \times \prod_{\substack{t=n+1 \\ t\neq m}}^{\infty} \frac{1 - (\lambda_m{}^2 - M^2)/(k^2t^2) - (s\lambda_m + Q)^2/(k^2t^4)}{1 - (\lambda_m{}^2 - M^2)/(k^2t^2)} .\end{aligned}$$

Convergence is now taken with m considered implicitly as a function of λ_m. Then

$$\lim km = (\lambda_m{}^2 - M^2)^{1/2},$$

$$\lim b_m/k = (s\lambda_m + Q)/(\lambda_m{}^2 - M^2)^{1/2},$$

and

$$\lim \frac{\sin((\pi/k)(\lambda_m{}^2 - M^2)^{1/2})}{\pi k(\lambda_m{}^2 - M^2)^{1/2}} = (-1)^{m+1} \frac{(s\lambda_m + Q)^2}{2(\lambda_m{}^2 - M^2)^2} .$$

The desired identity results from the definition of the integral as the limit of a sum.

Reference

1. L. de Branges, "Espaces Hilbertiens de Functions Entières," Masson, Paris, 1972.